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AVERAGING IN RANDOMLY PERTURBED MULTIFREQUENCY NON-LINEAR SYSTEMS†

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A procedure is proposed for separating motions in perturbed systems that are reducible to standard form with several fast phases. Non-resonant and resonant cases are considered. Systems with a hierarchy of phase rotation velocities are investigated. The slow motion is shown to converge to a diffusion process. An example is considered, namely, the perturbed motion of a gyroscope in gimbals.

1. We shall study systems with dynamics described by the equations

$$x^{\cdot}=\varepsilon F(x,\theta,\xi(t))+\varepsilon^2 G(x,\theta),\quad x\in R_n \tag{1.1}$$

$$\theta^{\cdot}=\omega(x)+\varepsilon H(x,\theta,\xi(t))+\varepsilon^2 D(x,\theta),\quad \theta\in R_m$$

$$F(x,\theta,\xi(t))=F_0(x,\theta)\xi(t),\quad H(x,\theta,\xi(t))=H_0(x,\theta)\xi(t) \tag{1.2}$$

where ε is a small parameter, $\xi(t)\in R_l$ is a stationary stochastic process with zero mean satisfying mixing conditions [1, 2], and F_0, H_0, G, D are matrices of the appropriate dimensions which are 2π-periodic functions of each of the components θ_k of the vector θ and sufficiently smooth [1, 2] as functions of their variables.

Special cases of system (1.1) were considered in [1, 2]. In [1] attention was devoted to a system with one fast phase ($m = 1$); conditions were formulated under which a stochastic averaging principle holds for such systems: as $\varepsilon\to 0$ the slow variable $x(t,\varepsilon)$ converges weakly [3] to a slow diffusion process $x_0(\tau)$, where $\tau=\varepsilon^2 t$. A similar result was established in [2] for multifrequency quasilinear systems with constant natural frequencies. We shall show here that a slow process also converges to a diffusion process in the more general system (1.1).

We will first consider the non-resonant case. Assume that the natural frequencies $\omega_k\geqslant\omega_{0k}>0$ satisfy the condition [4, 5]

$$|(\lambda,\ \omega)|\geqslant\delta(\lambda)>0 \tag{1.3}$$

for any vector λ with integer components, for all $x\in S$, where S is a bounded sphere in R_n.

Let us also assume that the right-hand sides of Eqs (1.1) satisfy the conditions of [1, 2]. As we recall, these conditions are satisfied if the functions F_0, H_0, G, D, ω are smooth enough and the stochastic process $\xi(t)$ satisfies mixing conditions, which are valid, e.g. for a normal stationary process.

We will denote the solution of system (1.1) by $x(t,\varepsilon)=x_\varepsilon(\tau)$ [1, 2] and define the limiting diffusion process $x_0(\tau)$ as the solution of the stochastic differential equation

$$dx_0=b(x_0)d\tau+\sigma(x_0)dw,\quad x_0(0)=x(0,\varepsilon)=r \tag{1.4}$$

with $\tau=\varepsilon^2 t$, $w(\tau)$ is a standard Wiener process, and the coefficients $b(x)$ and $\sigma(x)$ are defined by the formulae

$$b(x)=b_1(x)+b_2(x)+g(x) \tag{1.5}$$

where

$$b_j(x)=\langle B_j(x,\theta)\rangle,\quad B_j(x,\theta)=\int_0^\infty B_j(x,\theta,u)du \tag{1.6}$$

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$$g(x) = \langle G(x,\theta)\rangle$$
$$B_1 = \mathrm{M}[F_x(x,\theta+\omega(x)u,\xi(t+u))F(x,\theta,\xi(t))] \tag{1.7}$$
$$B_2 = \mathrm{M}[F_\theta(x,\theta+\omega(x)u,\xi(t+u))H(x,\theta,\xi(t))]$$

$$\sigma(x)\sigma'(x) = a(x); \quad a_{ij}(x) = \langle A_{ij}(x,\theta)\rangle \tag{1.8}$$
$$A_{ij}(x,\theta) = \int_{-\infty}^{\infty} A_{ij}(x,\theta,u)du$$
$$A_{ij}(x,\theta,u) = \mathrm{M}[F^i(x,\theta+\omega(x)u,\xi(t+u))F^j(x,\theta,\xi(t))]$$

Throughout, angular brackets denote averaging over space

$$\langle f(x,\theta)\rangle = (2\pi)^{-m}\int_0^{2\pi} f(x,\theta)d\theta = (2\pi)^{-m}\int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} f(x,\theta)d\theta_m$$

and F^i is the ith component of the vector F. It is assumed that spatial averages and the expressions (1.6) and (1.8) exist for all $x \in S$.

The following extension of the results of [1, 2] is easy to prove.

Theorem 1. Let the coefficients of the perturbed system (1.1) satisfy the above conditions. Then, if $\varepsilon \to 0$ and $\tau \in [0, L]$, the solution $x_\varepsilon(\tau)$ converges weakly to the diffusion process $x_0(\tau)$ defined by Eq. (1.4).

Theorem 1 is an obvious extension of analogous statements in [1, 2], and its proof may be omitted.

Remark. If the second-order coefficients depend on random perturbations and have the form $G(x, \theta, \xi(t))$, where

$$\mathrm{M}G(x, \theta, \xi(t)) = g(x, \theta) \tag{1.9}$$

then

$$g(x) = (2\pi)^{-m}\int_0^{2\pi} g(x,\theta)d\theta \tag{1.10}$$

In particular, if $MG = 0$, then $g(x) = 0$.

If the first-order coefficients have the form

$$F(x, \theta, \xi(t)) + f(x, \theta), \quad H(x, \theta, \xi(t)) + h(x, \theta) \tag{1.11}$$

where

$$\int_0^{2\pi} f(x,\theta)d\theta = 0 \tag{1.12}$$

then Theorem 1 remains valid, but an additional term occurs in the coefficient (1.5), corresponding to the second approximation of the averaging method for deterministic systems. In the general case this coefficient is rather complicated and it will not be written out, but a simpler special case will be used in Section 4.

2. An implicit assumption in problem (1.1) is that all the frequencies $\omega_k(x)$, irrespective of whether they are incommensurable or not, are quantities of the same order of magnitude, i.e. $|\omega_r(x)/\omega_l(x)| = O(1)$ for any $r, l = 1, \ldots, m$ when $x \in S$. In actual multidimensional systems the frequencies may differ considerably, so that for certain groups of frequencies $\omega_r(x)$ and $\omega_l(x)$ ($r = 1, \ldots, R$, $l = R + 1, \ldots, m$) we have $|\omega_r(x)/\omega_l(x)| = O(\varepsilon)$. When the frequencies exhibit such strong incommensurability, the equations of motion depend on both slowly and rapidly rotating phases. In the general case the equations may be written as

$$x^\cdot = \varepsilon F(x,\varphi,\theta,\xi(t)), \quad \varphi^\cdot = \varepsilon\Omega(x) + \varepsilon\Phi(x,\varphi,\theta,\xi(t))$$
$$\theta^\cdot = \omega(x) + \varepsilon H(x,\varphi,\theta,\xi(t)) \tag{2.1}$$

terms ($O(\varepsilon^2)$ are not written out). It is assumed that $\varphi \in R_k$, the vector Φ has structure (1.2) and satisfies the same conditions as the vector H, while the "slow" frequencies $\Omega(x)$ satisfy the incommensurability conditions (1.3) and are sufficiently smooth functions of x.

Systems with hierarchies of frequencies arise both because of the varying scales of the physical parameters of objects, or because of formal transformations of the equations of motion, e.g. when one is investigating resonant regimes [4, 5]. The basic features of the theory of deterministic systems are due to the fact that the slow variable x and slow phase φ vary at the same rate, i.e. the right-hand sides of (2.1) are not averaged with respect to φ and the dimensions of the vector of slow variables are increased.

In (2.1) it is assumed that $M\xi = 0$, i.e. $MF = 0$. This means [6] that over a time interval of length $O(\varepsilon^{-1})$ the variable x remains within a small neighbourhood of the initial position, but φ turns out to be a fast phase relative to x. This means that it is possible to average with respect to φ and to separate out the variable x as $\varepsilon \to 0$.

The possibility of successive averaging in deterministic systems with frequency hierarchies has been discussed [7]. We have investigated a stochastic system that can be reduced to the form (2.1) with two phases and a constant fast rotation frequency [8]. This special case will enable us to demonstrate the possibility of averaging with respect to the slow phase φ.

We shall show how to implement the principle of separation of motions in system (2.1). Consider a slow diffusion process $x_0(\tau)$, corresponding to a generating differential operator

$$L_0 = \beta'(x)\partial/\partial x + \tfrac{1}{2}\operatorname{Tr}\alpha(x)\partial^2/\partial x^2$$
$$\alpha(x) = \langle a(x,\varphi)\rangle, \quad \beta(x) = \langle b(x,\varphi) + k(x,\varphi)\rangle \tag{2.2}$$

The matrix $a(x, \varphi)$ is calculated by (1.8), with the variable x replaced by x, φ. Similarly, $b(x, \varphi)$ is the coefficient calculated by (1.6) and (1.7) with x replaced by x, φ. The coefficient $k(x, \varphi)$ is associated with the appearance of the additional variable φ

$$k(x,\varphi) = \langle K(x,\varphi,\theta)\rangle, \quad K(x,\varphi,\theta) = \int_0^\infty K(x,\varphi,\theta,u)du$$
$$K(x,\varphi,\theta,u) = \mathrm{M}[F_\varphi(x,\varphi,\theta+\omega(x)u,\xi(t+u))\Phi(x,\varphi,\theta,\xi(t))]$$

Theorem 2. Assume that system (2.1) satisfies the conditions listed above. Assume in addition that the operator (2.2) is uniformly parabolic and that the corresponding diffusion process $x_0(\tau)$ is regular. Then, if $\varepsilon \to 0$, $\tau \in [0, L]$, the solution $x_\varepsilon(\tau)$ of system (2.1) converges weakly to the process $x_0(\tau)$ with generating operator (2.2).

Proof. Note that when $\omega(x) = 1$ system (2.1) is precisely that studied in [8]. The proof will not be carried out in full detail, as it duplicates that presented in [1].

Consider the vector of slow variables x, $\varphi = y$, where we have put

$$x_\varepsilon(\tau) = x, \quad \varphi_\varepsilon(\tau) = \varphi, \quad y_\varepsilon(\tau) = y, \quad \tau = \varepsilon^2 t$$

A sufficient condition for $x_\varepsilon(\tau)$ to converge weakly to $x_0(\tau)$ is that [3]

$$\mathrm{M}_{y,\tau} f(x_\varepsilon(T)) \to \mathrm{M}_{x,\tau} f(x_0(T)), \quad \varepsilon \to 0 \tag{2.3}$$

for any function $f(x) \in C_4$ with $x \in S$, $\varphi \in R_k$, $\tau \in [0, T]$, $T \leqslant L$.

To prove (2.3), we use an auxiliary result that follows directly from a result in [8, Section 3]. Repeating the arguments of [8], except that instead of averaging with respect to time we average with respect to phases (as done in Section 1), we conclude that as $\varepsilon \to 0$ the slow process $y_\varepsilon(\tau)$ converges weakly to the $(n + k)$-dimensional diffusion process $y_{0\varepsilon}(\tau) = x_{0\varepsilon}(\tau)$, $\varphi_{0\varepsilon}(\tau)$ corresponding to the generating differential operator

$$L = \varepsilon^{-1}\Omega'(x)\partial/\partial\varphi + [b'(x,\varphi) + k'(x,\varphi)]\partial/\partial x + \kappa'(x,\varphi)\partial/\partial\varphi +$$
$$+ \tfrac{1}{2}\operatorname{Tr}[a(x,\varphi)\partial^2/\partial x^2 + d(x,\varphi)\partial^2/\partial x\partial\varphi + \delta(x,\varphi)\partial^2/\partial\varphi^2] \tag{2.4}$$

The coefficients b, k and a in (2.4) were defined previously; the coefficients κ, d and δ are defined similarly, but as we shall soon see, their precise form is immaterial. It is obvious that all the coefficients are periodic in φ; we shall assume in addition that the means of these functions with respect to φ exist for all $x \in S$.

Let us explain our assertion of weak convergence. Let

$$\mathrm{M}_{y,\tau}f(x_\varepsilon(T))=V_\varepsilon(y,\tau),\quad \mathrm{M}_{y,\tau}f(x_{0\varepsilon}(T))=V(y,\tau)$$

where $V(y, \tau) = V(x, \varphi, \tau)$ is defined as a solution of the following equation [3]

$$\partial V/\partial\tau + LV = 0,\quad V(x, \varphi, T) = f(x) \tag{2.5}$$

To say that x_ε converges weakly to $x_{0\varepsilon}$ means that, for all $x \in S$, $\varphi \in R_k$ and sufficiently small ε

$$|V_\varepsilon(y,\tau)-V(y,\tau)|\leqslant C\varepsilon \tag{2.6}$$

where C is a constant independent of ε. As in [8], it can be proved that the estimate (2.6) holds over a time interval in which the solution of Eq. (2.5) exists and is uniformly bounded (in the space $H_{4,2}$), and the functional $V_\varepsilon(y, \tau)$ exists and is uniformly bounded for sufficiently small ε and all $x \in S$, $\varphi \in R_k$.

To prove the existence of a solution of Eq. (2.5) and estimate it, we shall use the averaging principle of [6]. Together with (2.5), let us consider the averaged equation

$$\partial V_0/\partial\tau+L_0V_0=0,\qquad V_0(x,T)=f(x) \tag{2.7}$$

where L_0 is the operator (2.2). It follows from the properties of the coefficients of system (2.1) that $\alpha(x) \in C_2$, $\beta(x) \in C_1$. If at the same time $f(x) \in C_4$ and L_0 is uniformly parabolic, then [9] a solution $V_0(x, \tau) \in C_{4,2}$ exists such that [6]

$$|V(x,\varphi,\tau)-V_0(x,\tau)|\to 0,\quad \varepsilon\to 0 \tag{2.8}$$

where $V(x, \varphi, \tau)$ is a solution of Eq. (2.5). The limit exists uniformly in $x \in S$, $\varphi \in R_k$ for all $\tau \in [0, T]$. Now it follows from the definition of the process $x_0(\tau)$ that [3]

$$V_0(x,\tau)=\mathrm{M}_{x,\tau}f(x_0(\tau)) \tag{2.9}$$

Comparing (2.6) and (2.8) and using (2.9), we conclude that condition (2.3) holds over a time interval $\tau \in [0, T]$ in which the functional $V_\varepsilon(y, \tau)$ is bounded. As in [8], it is easy to prove that this estimate holds for any finite $T \leqslant L$, provided that $x_0(\tau)$ is a regular process.

Remark. It is obvious that additional terms of order ε^2 on the right of (2.1) will lead to the appearance of additional terms in the drift coefficient.

3. We will now study the effect of random perturbations on dynamical processes in the near-resonant region, accompanied by violation of condition (1.3). Resonance phenomena in deterministic systems have been investigated fairly thoroughly [4, 5]; in particular, the existence has been established of a "jamming" effect of solutions near the resonance surface, where the latter is defined by

$$(\lambda,\omega(x))=0,\quad |\lambda|^2\neq 0 \tag{3.1}$$

Let us investigate the effect of random perturbations on the motion near the resonance surface. We again consider a system of type (1.1)

$$x^\cdot=\varepsilon F(x,\theta,\xi(t)),\quad \theta^\cdot=\omega(x)+\varepsilon H(x,\theta,\xi(t)) \tag{3.2}$$

where the vectors F and H are of the form (1.2). Let us assume that the system has one resonance surface satisfying (3.1). As in the deterministic case, we introduce a new variable

$$\varphi=(\lambda,\theta)=(\Lambda,\psi)+\lambda_m\theta_m \tag{3.3}$$

where λ_k are the components of the vector λ, $\Lambda_k = \lambda_k$, $\psi_k = \theta_k$, $k = 1, \ldots, m-1$, $\lambda_m \neq 0$. Obvious algebra reduces (3.2) to the following form [4, 5]

$$x^\cdot=\varepsilon X(x,\varphi,\psi,\xi(t)),\quad \varphi^\cdot=(\lambda,\omega(x))+\varepsilon\Phi(x,\varphi,\psi,\xi(t)) \tag{3.4}$$

$$\psi^{\cdot} = \Omega(x) + \varepsilon\Psi(x, \varphi, \psi, \xi(t))$$

where $X = X_0\xi(t)$, $X_0 = F_0(x, \psi, \lambda_m^{-1}[\varphi - (\Lambda, \psi)])$ and so on, $\Omega_k(x) = \omega_k(x)$, $k = 1, \ldots, m-1$. Remembering that we are considering motion in the neighbourhood of the resonance surfaces, we introduce a new variable $q(t, \varepsilon)$ by the formula [4, 5]

$$(\lambda, \omega(x)) = \mu q, \quad \mu = \varepsilon^{1/2} \tag{3.5}$$

and transform (3.4) to a form involving new fast and slow variables

$$y^{\cdot} = \mu^2 Y(y, \varphi, \psi, \mu q, \xi(t)), \quad q^{\cdot} = \mu Q(y, \varphi, \psi, \mu q, \xi(t)) \tag{3.6}$$

$$\varphi^{\cdot} = \mu q + \mu^2 \Phi(y, \varphi, \psi, \mu q, \xi(t)), \quad \psi^{\cdot} = \Omega(y, \mu q) + \mu^2 \Psi(y, \varphi, \psi, \mu q, \xi(t))$$

where $y_k = x_k$, $k = 1, \ldots, m-1$; the coefficients of this system are obtained by appropriate transformations in (3.4) with due attention to the change of variable (3.5) (see [5]). The functions on the right of (3.6) are periodic in the phases φ and ψ. It follows from the conditions of Section 1 that $\Omega > 0$, and the condition that the resonance surface should not be cut implies that $q \neq 0$ (to fix our ideas, let $q > 0$). Consequently, the conclusions of Section 2 are applicable to system (3.6).

There are four different time-scales in Eqs (3.6): a fast phase ψ, a slow phase φ varying in the time scale μt, a slow process $q(t)$ with time-scale $\mu^2(t)$ and a slow process $y(t)$ with time-scale $\mu^4 t$ (the last two follow obviously from the condition $MY = MQ = 0$, which in turn follows from the forms of the functions Y and Q). Thus the variable y may be assumed constant over a time interval $O(\mu^{-2})$: $y(t, \mu) = y(0, \mu) = y_0$.

Let us rewrite (3.6) retaining only the terms essential for the analysis

$$q^{\cdot} = \mu Q_0(\varphi, \psi, \xi(t)) + \ldots, \quad \varphi^{\cdot} = \mu q + \ldots, \quad \psi^{\cdot} = \Omega_0 + \mu\Omega_1 q + \ldots$$
$$Q_0 = Q(y_0, \varphi, \psi, 0, \xi(t)), \quad \Omega_0 = \Omega(y_0, 0), \quad \Omega_1 = \Omega_y(y_0, 0) \tag{3.7}$$

Terms of order μ^2 are omitted, as their means vanish (cf. the remark at the end of Section 1). System (3.7) is obviously a special case of (2.1).

Applying the conclusions of Section 2 to (3.7), we conclude that the slow variable $q(t, \varepsilon)$ is approximated over a time interval of length $O(\mu^{-2})$ by a slow diffusion process $q_0(s)$ satisfying the equation

$$dq_0 = \sigma dw, \quad s = \mu t = \varepsilon^{1/2} t \tag{3.8}$$

where

$$\sigma^2 = \langle D(\varphi, \psi) \rangle, \quad D(\varphi, \psi) = \int_{-\infty}^{\infty} MQ_0(\varphi, \psi + \Omega_0 u, \xi(t+u)) Q_0(\varphi, \psi, \xi(t)) du \tag{3.9}$$

The angular brackets denote averaging over phases; that D is independent of t follows from the fact that $\xi(t)$ is a stationary process and from the form of the function Q_0.

Thus, $q_0(s)$ is a normal process with zero mean and variance $D(s) = \sigma^2 s$. This result implies that, over a time interval of length $O(\varepsilon^{-1})$, the trajectory of the perturbed system remains in a small $O(\varepsilon^{-1/2})$ neighbourhood of the resonance surface, but, because of the increase in the variance, it tends to leave that neighbourhood. The conclusion is especially easy to interpret in the problem of forced rotations of a single-frequency non-linear system, when relation (3.1) is

$$\lambda_1 \omega(x) + \lambda_2 \omega_0 = 0 \tag{3.10}$$

where $\omega(x)$ is the natural frequency of the system and ω_0 is that of the perturbation. It follows from our discussion that this system cannot sustain a steady rotatory regime with a frequency $\omega = -\lambda_2\lambda_1^{-1}\omega_0$. The frequency fluctuations are determined by Eqs (3.8) and (3.9).

If we retain higher-order terms in the equations of motion, we must put

$$x^{\cdot} = \varepsilon F(x, \theta, \xi(t)) + \varepsilon^{3/2} G(x, \theta) \tag{3.11}$$

$$\theta^{\cdot} = \omega(x) + \varepsilon H(x, \theta, \xi(t)) + \varepsilon^{3/2} D(x, \theta)$$

where the terms G and D have the same meaning as in (1.1). Then the transformed version of Eq. (3.7) will be

$$q^{\cdot} = \mu Q_0(\varphi, \psi, \xi(t)) + \mu^2 Q_1(\varphi, \psi)$$
$$\varphi^{\cdot} = \mu q, \quad \psi^{\cdot} = \Omega_0 + \mu\Omega_1 q \tag{3.12}$$

where Q_1 is an additional deterministic term, obtained by transforming the function G. Equations (3.8) now involve an additional drift coefficient

$$dq_0 = b\,ds + \sigma dw, \quad b = \langle Q_1(\varphi, \psi)\rangle \tag{3.13}$$

In that case $q_0(s)$ is a normal process with mean $m(s) = q(0) + bs$ and variance (3.9).

4. To illustrate this, let us consider a perturbed rotating gyroscope. Previous publications [5] have studied the motion of a balanced gyroscope in gimbals acted upon by a small periodic torque; the existence of stable periodic motions has been established. Attention has also been given [10] to oscillations of an unbalanced gyroscope acted upon by a random torque of the white-noise type, and the domains of existence of steady-state solutions have been studied.

Let us consider the motion of an unbalanced gyroscope in gimbals mounted on an oscillating base. Suppose that a small periodic torque and a small torque due to friction forces are applied to the axis of the internal gimbal. We shall study the effect of random oscillations of the base on the rotation of the internal gimbal and the mean drift velocity of the gyroscope.

We write the equations of motion as [5, 10]

$$\theta'' + \varepsilon u(\theta) = \varepsilon(k \sin\Omega t + \beta\xi(t)\cos\theta) - \varepsilon^2 \nu\theta' \tag{4.1}$$

$$y = m(l - \cos\theta)(1 - \kappa\cos^2\theta)^{-1} \tag{4.2}$$

$$u(\theta) = dU/d\theta, \quad 2U(\theta) = -\beta\cos\theta + (l - \cos\theta)^2[(1 - \kappa\cos^2\theta)]^{-1} \tag{4.3}$$

where θ is the angle of rotation of the internal gimbal, y is the drift velocity of the gyroscope, the prime denotes differentiation with respect to the dimensionless "nutational" variable t, and l, κ, m and β are constants expressed in terms of the inertial and kinetic parameters of the gyroscope and the gimbals [5, 10]. The small parameter ε on the right of (4.1) characterizes the smallness of the perturbing and dissipative factors, while the small parameter on the left means that the dimensionless velocity of rotation of the internal gimbal θ' is large compared with the dimensionless drift velocity y of the gyroscope, and the rotation of the gimbal is nearly uniform. The stochastic process $\xi(t)$ defines the acceleration of the base; it is assumed that $\xi(t)$ is a stationary process with zero mean, correlation function $K(t)$ and spectral density $S(\omega)$.

The energy integral of the conservative system corresponding to the right-hand side of (4.1) is

$$(\theta')^2 + 2\varepsilon U(\theta) = x^2 \tag{4.4}$$

To apply the asymptotic method, we treat (4.4) as a change of variables. Using the fact that ε is small, we can write

$$\theta' = x - \varepsilon U(\theta)x^{-1} + \varepsilon^2 \ldots \tag{4.5}$$

Substituting (4.5) into (4.1), we obtain a system of equations

$$x' = \varepsilon(1 - \varepsilon U(\theta)x^{-2})(k\sin\psi + \beta\xi(t)\cos\theta) - \varepsilon^2\nu\theta' - \varepsilon^2 x^{-2}u(\theta)U(\theta)$$
$$\theta' = x - \varepsilon U(\theta)x^{-1} + \varepsilon^2 \ldots, \quad \psi' = \Omega \tag{4.6}$$

In the non-resonant case ($x \neq \Omega$) the process $x(t, \varepsilon)$ converges weakly to a slow diffusion process (1.4). In view of (4.6), the coefficients of Eq. (1.4) may be written as

$$b_2 = 0, \quad b_1 = \beta^2 S_x(x)/4 = a_x(x)/2, \quad a(x) = \beta^2 S(x)/2 \tag{4.7}$$

It follows from (1.4) and (4.7) that the probability density $p(x, \tau)$ of the process $x_0(\tau)$ is determined by the equation [3]

$$\frac{\partial p}{\partial \tau}+\frac{\partial}{\partial x}\left[\left(-\nu x+\frac{1}{2}\frac{da}{dx}\right)p\right]-\frac{1}{2}\frac{\partial^2}{\partial x^2}(ap)=0 \tag{4.8}$$

This equation has a steady-state solution

$$p(x)=C\exp\left[-2\int_0^x \frac{\nu z}{a(z)}\right]dz \tag{4.9}$$

where the coefficient C is determined by normalization. If $\xi(t)$ is white noise with a fixed spectral density S_0, then

$$p(x)=C\exp[-\nu x^2/a], \quad a=\beta^2 S_0/4 \tag{4.10}$$

i.e. $x_0(\tau)$ is a normal stationary process. Clearly, a steady-state solution exists only when $\nu > 0$.

We now investigate the perturbed resonance regime. To allow for the effect of dissipation, let us write the equation of motion as

$$\theta''+\mu^2 u(\theta)=\mu^2(k\sin\Omega t+\beta\xi(t)\cos\theta)-\mu^3\nu\theta', \quad \mu^2=\varepsilon \tag{4.11}$$

Using the change of variable (4.5) and putting $x-\Omega=\mu q$, $\theta-\psi=\varphi$, we write

$$q'=\mu(k\sin\psi+\beta\xi(t)\cos(\varphi+\psi))-\mu^2\nu\theta', \quad \varphi'=\mu q, \quad \psi'=\Omega \tag{4.12}$$

It follows from (4.10), (3.12) and (3.13) that the perturbation $q(t,\mu)$ converges weakly to a diffusion process (3.13) with drift coefficient $-\Omega\nu$ and variance (3.9) with coefficient $\sigma^2=a(\Omega)=\beta^2 S(\Omega)/2$.

The average drift velocity of the gyroscope

$$\langle y\rangle=\frac{m}{2\pi}\int_0^{2\pi}(l-\cos\theta)(1-\kappa\cos^2\theta)^{-1}d\theta \tag{4.13}$$

in the resonant and non-resonant cases is determined just as in the deterministic problem; it does not depend on the perturbing factors.

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